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Possible primitive notions for geometry of spine spaces

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ABSTRACT

New systems of notions specific to the geometry of spine spaces, are introduced. In particular parallelism turns out to be a sufficient primitive notion to express the geometry of a spine space, and we show that structures related to projective closure are definitionally equivalent to spine spaces.

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1. Motivations and results

In this paper we are trying to answer fundamental questions concerning possible systems of primitive notions for the geometry of spine spaces. The geometry in question, introduced in [16], generalizes the geometry of slit spaces, affine Grassmannians, and the geometry of the structure of linear complements (cf. [19]); in a sense, the construction of a spine space embedded in a projective Grassmannian resembles the construction of an affine space embedded in a projective space. While for classical geometries the problem to find adequate systems of relatively simple primitive notions and even to axiomatize them in the language of those notions has been already solved, there is no such system for spine spaces.

Formally, a spine space \mathfrak{A} is a partial linear space equipped with a (partial) parallelism, so the class of its lines is divided into two classes: affine lines and projective lines. Besides, projective lines may be of two sorts. Are all these notions necessary to develop the geometry of spine spaces?

Some answers are already given in [16], quoted in 3.1(iv): roughly speaking, the parallelism is definable in terms of the incidence of points and lines, and the class of affine lines is definable in terms of the class of projective lines, provided that the latter is nonvoid. Clearly, when developing the geometry of a partial linear space we can use the language with the ternary relation of collinearity of points instead of the language of incidence. Then, in the case when the adjacency (binary collinearity) of points is nontrivial, i.e. when \mathfrak{A} is not a linear space, a common question arises if this adjacency can be used as a primitive notion. The answer is positive, excluding spine spaces of some specific type (cf. 4.2 quoted from [14]).

Another question, which is standard in the foundations of linear geometries, is the following: can we develop a particular geometry as a theory whose individuals are lines? Since the ternary concurrency relation on lines is sufficient for this purpose, our question is: can we define the concurrency in terms of binary adjacency of lines (i.e. the relation of line intersection), that is, can we use the adjacency of lines as a primitive notion in our geometry of spine spaces? The affirmative answer to this question is given in Proposition 4.4.

In this paper some new incidence systems are proposed, which resemble the projective closure of \mathfrak{A} and which are also definitionally equivalent to \mathfrak{A} (cf. Section 4.2).

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By suitable interpretations we obtain that the geometry of spine spaces can be developed as a theory of affine partial linear spaces of some kind, i.e. as a theory of incidence with affine lines only, but equipped with parallelism.

On the universe of points of a spine space one can introduce two other structures of lines which determine the same adjacency of points as in the original case; roughly speaking these “new” lines are maximal linear subspaces of some types. Therefore, it appears naturally to ask if one can develop the geometry of spine spaces in terms of the adjacency of these new lines.

Our results need some additional assumptions which state that \mathfrak{A} does not degenerate in some ways. For readers' convenience we have gathered respective assumptions in Table 1. Connections with classical geometries and relevant results are discussed in Appendix A.

We have tried to clothe our reasonings in the style of classical synthetic geometry, despite the fact that formally a spine space is defined in the language of linear algebra. Thanks to characterizations like the one in [3] we can view a Grassmann space as a union of some projective spaces. Similarly, we can imagine a spine space as a union of slit spaces (projective or affine spaces in the extremes). This approach lets us avoid analytical techniques and replace them by proofs based on visual geometrical ideas. Another one of our goals was to point out that the geometry of spine spaces is a “real” geometry.

2. Models

Every partial linear space $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ determines the *adjacency* relation as follows: we say that two points a, b of \mathfrak{M} are *adjacent* and write $a \sim b$ iff they are on a line of \mathfrak{M} . If that is the case we write $\overline{a, b}$ for the line which joins a and b . Two lines of \mathfrak{M} are adjacent if they have a common point. In case \mathfrak{M} is equipped with a (partial) parallelism \parallel we say that a line l of \mathfrak{M} is *affine* iff $l \parallel l$ (cf. [20]). Nonaffine lines are then frequently called *projective*.

We begin with Grassmannian geometries, i.e. with geometries defined on the universe $\text{Sub}_k(\mathbb{V})$ of all k -dimensional subspaces of a fixed vector space \mathbb{V} . Unless explicitly stated otherwise, \mathbb{V} is of any, possibly infinite, dimension. On such a universe one has a natural structure of a partial linear space called a *space of pencils*

$$\mathfrak{P} = \mathbf{P}_k(\mathbb{V}) = \langle \text{Sub}_k(\mathbb{V}), \mathcal{P} \rangle, \quad (1)$$

where \mathcal{P} stands for the family of all k -pencils, that is, the sets of the form

$$[H, B]_k = \{U \in \text{Sub}_k(\mathbb{V}) : H \subset U \subset B\}$$

with $H \in \text{Sub}_{k-1}(\mathbb{V})$, $B \in \text{Sub}_{k+1}(\mathbb{V})$, and $H \subset B$. We say that a subspace of a partial linear space is *strong* if every two of its points are adjacent. The maximal strong subspaces of the structure $\mathbf{P}_k(\mathbb{V})$ are the maximal *tops*, i.e. sets $T(B) = \text{Sub}_k(B)$, where $B \in \text{Sub}_{k+1}(\mathbb{V})$ and the maximal *stars*, i.e. the sets of the form $S(H) = \text{Sup}_k(H)$, where $H \in \text{Sub}_{k-1}(\mathbb{V})$ and $\text{Sup}_k(H)$ stands for the set of all k -dimensional subspaces of \mathbb{V} which contain H . Let us write \mathcal{T} for the family of all maximal tops and \mathcal{S} for the family of all maximal stars of $\mathbf{P}_k(\mathbb{V})$.

Structures under further consideration are (mainly) *spine spaces*

$$\mathfrak{A} = \mathbf{A}_{k,m}(\mathbb{V}, W) = \langle \mathcal{F}_{k,m}(W), \mathcal{L}, \parallel \rangle, \quad (2)$$

where W is a fixed subspace of \mathbb{V} , $\mathcal{F}_{k,m}(W)$ stands for the set of all k -dimensional subspaces U of V with $\dim(U \cap W) = m$, and \mathcal{L} is the set of all nontrivial (at least two element) sections

$$L = p \cap \mathcal{F}_{k,m}(W) \quad (3)$$

where $p \in \mathcal{P}$. The necessary assumptions on k and m are given in Table 1, in conditions (i)–(iv). The relation \parallel is a partial parallelism in \mathfrak{A} . Spine spaces were introduced in [16].

There is a detailed classification of lines and strong subspaces of spine spaces in [17] and [14]. A line L of \mathfrak{A} of the form (3) is either projective (then $p \subset \mathcal{F}_{k,m}(W)$) or affine (then $|p \setminus \mathcal{F}_{k,m}(W)| = 1$). So, an affine line L has its improper point (point at infinity) L^∞ , formally $\{L^\infty\} = p \setminus \mathcal{F}_{k,m}(W)$. A projective line L can be of one of two sorts α or ω , what we write respectively as $L \in \mathcal{L}^\alpha$ or $L \in \mathcal{L}^\omega$.¹ In case L is affine we write that $L \in \mathcal{A}$. To shorten notation we adopt a convention that $\{\sigma, -\sigma\} = \{\alpha, \omega\}$.

Strong subspaces of \mathfrak{A} are restrictions of strong subspaces of $\mathbf{P}_k(\mathbb{V})$ to the point set of \mathfrak{A} . Consequently, maximal strong subspaces of \mathfrak{A} are appropriate restrictions of stars and tops of $\mathbf{P}_k(\mathbb{V})$. Maximal strong subspaces of \mathfrak{A} are projective spaces or slit spaces (cf. [11,20]). Accordingly, the first class will be denoted by \mathcal{P}^σ , and the other by \mathcal{H}^σ . A subspace from \mathcal{P}^σ , called σ -*projective*, has all its lines of sort σ , while a subspace from \mathcal{H}^σ , called σ -*semiaffine*, contains affine lines and its projective lines are all of sort σ . We sometimes say shortly *projective* or *semiaffine* when we do not care about the sort. Semiaffine subspaces with no projective lines are called *affine*.

For suitable restrictions of \mathfrak{A} we write

¹ Precise analytical definitions of respective types and sorts of lines, planes or subspaces in general are given in [16,17,14]. We do not quote them here because only synthetic properties of these classes presented in the paper are necessary to follow the reasonings.

$$\mathfrak{A}^\alpha := \langle \mathcal{F}_{k,m}(W), \mathcal{A} \cup \mathcal{L}^\alpha \rangle, \quad \mathfrak{A}^\omega := \langle \mathcal{F}_{k,m}(W), \mathcal{A} \cup \mathcal{L}^\omega \rangle, \quad \text{and} \quad \mathfrak{A}^* := \langle \mathcal{F}_{k,m}(W), \mathcal{L}^\alpha \cup \mathcal{L}^\omega \rangle. \quad (4)$$

The parallelism \parallel is defined on the family \mathcal{A} by the requirement: $L_1 \parallel L_2$ iff $L_1^\infty = L_2^\infty$ (this relation is also *intrinsically* definable in terms of the incidence geometry of \mathfrak{A} , cf. [16]). The set of equivalence classes of this parallelism yields the point set of the *horizon* \mathfrak{A}^∞ of \mathfrak{A} . Recall that the horizon of $\mathbf{A}_{k,m}(\mathbb{V}, W)$ is (up to an isomorphism, cf. [16]) again a spine space, namely $\mathbf{A}_{k,m+1}(\mathbb{V}, W)$ with its affine lines deleted. A line L of \mathfrak{A}^∞ is the set of improper points of the lines which lie on a *plane* A of \mathfrak{A} (cf. [16,18]); the term *plane* in \mathfrak{A} means a plane contained in a strong subspace of \mathfrak{A} . More specifically, if A is σ -semiaffine (i.e. A is contained in an element of \mathcal{H}^σ), then the sort of L in \mathfrak{A}^∞ is $-\sigma$. Thus we sometimes call an affine subspace σ -*affine* when its horizon is of sort $-\sigma$ on \mathfrak{A}^∞ .

In ordinary affine geometry (which is a very particular case of spine geometry) completing lines by their improper points leads to a projective space, and the improper points yield a projective space. In the general spine geometry various restrictions of this procedure are possible if one considers substructures of a spine space \mathfrak{A} obtained by restricting its line set to the lines of fixed sorts (cf. (4)). Through the definitions (7) we introduce these possible particular “closures”.

In the sequel we use notation where subscript 0 refers to \mathfrak{A} , and subscript 1 to the horizon of \mathfrak{A} , specifically we put:

$$\mathcal{F}_0 := \mathcal{F}_{k,m}(W), \quad \mathcal{F}_1 := \mathcal{F}_{k,m+1}(W), \quad (5)$$

and distinguish three classes of lines:

\mathcal{L}_0^τ – affine lines of \mathfrak{A} , each one completed with its improper point;
 \mathcal{L}_0^σ – σ -projective lines of \mathfrak{A} ;
 \mathcal{L}_1^σ – σ -projective lines of the horizon of \mathfrak{A} .

Similarly, we have $\mathcal{P}_0^\sigma, \mathcal{P}_1^\sigma$ – projective maximal strong subspaces, and $\mathcal{H}_0^\sigma, \mathcal{H}_1^\sigma$ – semiaffine maximal strong subspaces in \mathfrak{A} and \mathfrak{A}^∞ respectively. Let us set

$$\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_1, \quad \widetilde{\mathcal{L}}^\alpha := \mathcal{L}_0^\tau \cup \mathcal{L}_0^\alpha \cup \mathcal{L}_1^\omega, \quad \widetilde{\mathcal{L}}^\omega := \mathcal{L}_0^\tau \cup \mathcal{L}_0^\omega \cup \mathcal{L}_1^\alpha. \quad (6)$$

Some remark is in order here. Note that $\mathcal{L}_0^\tau \neq \emptyset$ all the time, moreover $\mathcal{L}_0^\sigma \neq \emptyset$ or $\mathcal{L}_1^{-\sigma} \neq \emptyset$ as otherwise \mathfrak{A} would be a linear space (cf. Table 1 in Section A.3). So it may happen that projective lines of some sort in $\widetilde{\mathcal{L}}^\alpha$ or in $\widetilde{\mathcal{L}}^\omega$ are missing, but both these sets are always nonvoid. Thus we define

$$\mathfrak{N}^\alpha := \langle \mathcal{F}, \widetilde{\mathcal{L}}^\alpha \rangle, \quad \mathfrak{N}^\omega := \langle \mathcal{F}, \widetilde{\mathcal{L}}^\omega \rangle, \quad \text{and} \quad \widetilde{\mathfrak{A}} := \langle \mathcal{F}, \widetilde{\mathcal{L}}^\alpha \cup \widetilde{\mathcal{L}}^\omega \rangle. \quad (7)$$

Clearly, the structures $\mathfrak{N}^\omega, \mathfrak{N}^\alpha$, and $\widetilde{\mathfrak{A}}$ are partial linear spaces. Slightly imprecisely we can say that \mathfrak{N}^σ is the *partial projective closure* of \mathfrak{A}^σ for $\sigma = \alpha, \omega$, and $\widetilde{\mathfrak{A}}$ is the closure of \mathfrak{A}^* . The statement is really formally imprecise since \mathfrak{A}^σ does not contain the parallelism, and this relation is essential in constructing the horizon. As we shall see further, \mathfrak{N}^σ is *not definable* within \mathfrak{A}^σ , so one can consider it as a suitable closure only by means of external definition referring to the whole structure \mathfrak{A} .

As we shall see, the parallelism \parallel , which is inessential in the geometry of \mathfrak{A} , as it is intrinsically definable, may play essential role in investigations on some restrictions and their closures. In particular, we consider the structure

$$\mathfrak{A}^\tau := \langle \mathcal{F}_{k,m}(W), \mathcal{A}, \parallel \rangle, \quad (8)$$

where \mathcal{A} consists of (yet uncompleted) affine lines of \mathfrak{A} .

Finally, on the universe $\mathcal{F}_{k,m}(W)$ we introduce another two structures of lines:

$$\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W) := \langle \mathcal{F}_{k,m}(W), \mathcal{T}_W \rangle \quad \text{and} \quad \mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W) := \langle \mathcal{F}_{k,m}(W), \mathcal{S}_W \rangle, \quad (9)$$

where \mathcal{T}_W consists of nontrivial sections of tops from \mathcal{T} with $\mathcal{F}_{k,m}(W)$, and \mathcal{S}_W consists of nontrivial sections of stars from \mathcal{S} with $\mathcal{F}_{k,m}(W)$. One can note that

$$\mathcal{S}_W \supset \mathcal{H}^\alpha \cup \mathcal{P}^\omega \quad \text{and} \quad \mathcal{T}_W \supset \mathcal{P}^\alpha \cup \mathcal{H}^\omega. \quad (10)$$

Most of the time, we can write “=” instead of “ \subset ” in (10), but in some degenerate cases we have $\mathcal{S}_W = \mathcal{H}^\alpha \cup \mathcal{L}^\omega$ or $\mathcal{T}_W = \mathcal{H}^\omega \cup \mathcal{L}^\alpha$. These two new structures can be viewed as spine space together with its tops and stars as lines. From (10) the following is immediate.

Fact 2.1. *The adjacency of points in \mathfrak{A} and in $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ (in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ resp.) coincide.*

To make our presentation more intuitive we shall substitute $k := k + 1$ and $m := m + 1$, and after that we shall restrict our investigations to $\mathfrak{M} = \mathbf{A}_{k+1,m+1}^{\text{star}}(\mathbb{V}, W)$ (for the structure of tops the reasoning runs dually). It is seen (cf. (9) and [17]) that the lines of \mathfrak{M} correspond to the elements of $\mathcal{F}_{k,m}(W) \cup \mathcal{F}_{k,m+1}(W)$. Two stars $S(U_1)$ and $S(U_2)$ ($U_1, U_2 \in \text{Sub}_k(\mathbb{V})$) have a common point in \mathfrak{M} iff $U_1 + U_2 \in \mathcal{F}_{k+1,m+1}(W)$; in such a case we write $U_1 \sim U_2$. This can be read as follows: the lines of \mathfrak{M} can be interpreted as the points of \mathfrak{A} and its horizon \mathfrak{A}^∞ , i.e. as the points of \mathfrak{N}^ω .

Fact 2.2. For distinct $U_1, U_2 \in \mathcal{F}$ we have: $U_1 \sim U_2$ iff U_1, U_2 are collinear in \mathfrak{N}^ω . In other words, the structure of adjacency of lines of $\mathbf{A}_{k+1,m+1}^{\text{star}}(\mathbb{V}, W)$ is isomorphic to the adjacency of points of \mathfrak{N}^ω .

Analogous relation holds between the adjacency of lines of $\mathbf{A}_{k-1,m}^{\text{top}}(\mathbb{V}, W)$ and the adjacency of \mathfrak{N}^α .

The main result of the paper, proved through Sections 3 and 4, can be read as follows

Theorem (Main). Let \mathfrak{A} be a spine space which is neither a linear space, nor a space of pencils. The geometry of \mathfrak{A} can be expressed in any of the following languages:

	System	Assumptions
(i)	points and their adjacency	for both $\sigma = \alpha, \omega$ either \mathfrak{A} does not contain a σ -projective line or it has a σ -projective maximal strong subspace.
(ii)	all lines and their adjacency	(none)
(iii)	all projective lines and their adjacency	\mathfrak{A} contains projective lines
(iv)	points, affine lines, and the parallelism	as in (i)
(v)	points (proper and improper) and the union of σ -projective lines, affine lines, and directions of σ -affine planes (as improper lines)	\mathfrak{A} contains both σ -affine and $(-\sigma)$ -affine planes
(vi)	points as in (v) and their adjacency w.r.t. the lines defined in (v)	\mathfrak{A} contains a σ -projective maximal strong subspace and a σ -affine plane
(vii)	points and stars	as in (i) with $\sigma = \alpha$ only
(viii)	points and tops	as in (i) with $\sigma = \omega$ only
(ix)	stars and their adjacency	both \mathfrak{A}^∞ and $(\mathfrak{A}^\infty)^\infty$ have ω -projective maximal strong subspaces
(x)	tops and their adjacency	as in (ix), with ω replaced by α

Proof. (i) follows from 4.2; (ii) and (iii) follow from 4.5; (iv) follows from 4.13; (v) follows from 4.15; (vi) follows from 4.16; (vii) and (viii) follow from 3.4 and 3.3; (ix) and (x) follow from 4.18 and 4.20. \square

3. Some elementary properties

3.1. Known results on interpretability

Some of the structures defined in Section 2 are already known to be equivalent (or non-equivalent) to the underlying spine space \mathfrak{A} , or such their property is easily derivable from the known results. Let us begin with gathering together these known results.

Fact 3.1. Let $\sigma \in \{\alpha, \omega\}$.

- (i) If $\mathcal{L}^\sigma \neq \emptyset$ then the class of affine lines is definable in terms of the adjacency of the structure $\langle \mathcal{F}_{k,m}(W), \mathcal{L}^\sigma \rangle$ [14, Prop. 4.7, Lem. 4.8]. Moreover, the class \mathcal{A} is distinguishable in $\mathcal{L}^\sigma \cup \mathcal{A}$ ([16, Lem. 3.5] and $\mathcal{H}^\sigma \neq \emptyset$ by (x), (xi) of Table 1) and therefore the structures \mathfrak{A}^σ and $\langle \mathcal{F}_{k,m}(W), \mathcal{L}^\sigma \rangle$ are definitionally equivalent.
- (ii) If $\mathcal{L}^\sigma \neq \emptyset$ then the class \mathcal{L}^σ is not definable in $\mathfrak{A}^{-\sigma}$ (following the idea of the proof of [14, Prop. 4.6]).
- (iii) If \mathfrak{A} contains projective lines then the class \mathcal{L} is not definable in the structure $\langle \mathcal{F}_{k,m}(W), \mathcal{A} \rangle$ (following the idea of the proof of [14, Prop. 4.6]).
- (iv) If $\mathcal{L}^\alpha \cup \mathcal{L}^\omega \neq \emptyset$ then affine lines and the parallelism of affine lines of \mathfrak{A} is definable in \mathfrak{A}^* [16, Cor. 3.4 and Cor. 3.7]. Consequently, the structures \mathfrak{A} and \mathfrak{A}^* are definitionally equivalent.
- (v) If $\mathcal{L}^\alpha \cup \mathcal{L}^\omega = \emptyset$ then the parallelism \parallel is definable in \mathfrak{A} [16, Cor. 3.7] and therefore the structures $\mathfrak{A} = \langle \mathcal{F}_{k,m}(W), \mathcal{A} \rangle$ and \mathfrak{A}^τ are definitionally equivalent.

As a consequence of 3.1(ii), (iii), the geometry of spine spaces cannot be expressed either as the geometry of projective lines of one sort only or as a pure incidence theory of affine lines only (if nonaffine lines exist).²

² The devil is in the details and therefore we must very carefully and exactly formulate details (that at first sight may seem only technical) of the assumptions and conclusions. Let us quote one example of a misleading handwaving argumentation: From \mathcal{L}^α we can define the class of affine lines. Moreover, we can also define affine planes, so the lines of \mathfrak{A}^∞ are definable. Consequently, \mathfrak{N}^α is definable from \mathfrak{A}^α and, in view of 4.8, \mathfrak{A} is definable from \mathfrak{A}^α . The point is, however, that to complete the above reasoning we must be able to define in \mathfrak{A}^α the parallelism of \mathfrak{A} , which is not possible. However, the relation “ $L_1 \parallel L_2$ and L_1, L_2 lie in one α -component of \mathfrak{A} ” (cf. [17]) is definable in \mathfrak{A}^α .

It is clear that $\tilde{\mathfrak{A}}$ is definable in \mathfrak{A} ; it is not so clear that \mathfrak{A}^α and \mathfrak{A}^ω are also definable.

Remark 3.2. If $\dim(W) \neq k$ or $\dim(\mathbb{V}) \neq 2k$ then the structures \mathfrak{A}^α and \mathfrak{A}^ω (and thus also \mathfrak{N}^α and \mathfrak{N}^ω) are definable in \mathfrak{A} . If $\dim(W) = k$ and $\dim(\mathbb{V}) = 2k$ then the pair $\{\mathfrak{A}^\alpha, \mathfrak{A}^\omega\}$ is definable in \mathfrak{A} (if additionally the ground field admits an anti-automorphism then $\mathfrak{A}^\alpha \cong \mathfrak{A}^\omega$ and $\mathfrak{N}^\alpha \cong \mathfrak{N}^\omega$).

Hint. It is obvious that the class of projective lines of \mathfrak{A} is definable and also the division of this class into two sorts is definable. Therefore one can define in \mathfrak{A} the pair $\{\mathfrak{A}^\alpha, \mathfrak{A}^\omega\}$. Assume that $\mathcal{L}^\alpha \neq \emptyset \neq \mathcal{L}^\omega$ (otherwise our claim is obvious). Maximal projective strong subspaces of \mathfrak{A}^σ are elements of \mathcal{P}^σ and maximal strong subspaces which contain affine lines are elements of \mathcal{H}^σ , so their dimensions are $k - m$ and $\dim(\mathbb{V}) - k$ resp. for $\sigma = \alpha$, and $\dim(W) - m$ and k for $\sigma = \omega$ (cf. [17]). The equalities of corresponding dimensions hold only iff $\dim(W) = k$ and $\dim(\mathbb{V}) = 2k$. If that is not the case, then these two structures are distinguishable in terms of the geometry of \mathfrak{A} . When $\dim(W) = k$ and $\dim(\mathbb{V}) = 2k$, and moreover \mathfrak{A} admits an automorphism derived from the correlation in the projective space over \mathbb{V} with W being a selfconjugate subspace, then this automorphism establishes an isomorphism of \mathfrak{A}^α onto \mathfrak{A}^ω and of \mathfrak{N}^α onto \mathfrak{N}^ω . \square

With a similar reasoning we obtain

Remark 3.3. If $\dim(W) \neq k$ or $\dim(\mathbb{V}) \neq 2k$ then the structures $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ and $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ are definable in \mathfrak{A} . If $\dim(W) = k$ and $\dim(\mathbb{V}) = 2k$ then the pair $\{\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W), \mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)\}$ is definable in \mathfrak{A} (if additionally the ground field admits an anti-automorphism then $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W) \cong \mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$).

Hint. The lines of $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ and $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ are maximal strong subspaces of \mathfrak{A} and each such a subspace X is determined by a proper triangle U_1, U_2, U_3 in \mathfrak{A} by the formula $X = [U_1, U_2, U_3]_{\sim}$; consequently the class of strong subspaces of \mathfrak{A} is elementarily definable in \mathfrak{A} . The classification of these subspaces into corresponding types and sorts goes as in 3.2. \square

Let us close this part with

Fact 3.4. The structure \mathfrak{A} is definable in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ ($\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$) iff $\mathcal{L}^\omega = \emptyset$ or $\mathcal{P}^\omega \neq \emptyset$ ($\mathcal{L}^\alpha = \emptyset$ or $\mathcal{P}^\alpha \neq \emptyset$).

Proof. Let us begin with $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$. In view of 2.1, from 4.2 or directly from [14, Cor. 4.11] it is obvious that \mathfrak{A} is definable in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ in all the cases except the two:

- (i) $\mathcal{L}^\omega \neq \emptyset = \mathcal{P}^\omega$ and
- (ii) $\mathcal{L}^\alpha \neq \emptyset = \mathcal{P}^\alpha$.

Suppose that (ii) holds; then the lines of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ are the elements of $\mathcal{L}^\alpha \cup \mathcal{H}^\omega$. From [14, Lem. 4.12] we infer that in terms of the adjacency of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ we can define the classes \mathcal{L}^ω and \mathcal{A} . The elements of \mathcal{H}^ω can be distinguished in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ as those “lines”, which contain a set in \mathcal{A} , so the class \mathcal{L}^α is also definable in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$. Thus to ensure that \mathfrak{A} is definable in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ it only remains to assume that (i) does not hold.

Suppose that (i) holds (but (ii) fails); then the line set of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ is $\mathcal{P}^\alpha \cup \mathcal{H}^\omega$. Again from [14, Lem. 4.12], in terms of the adjacency of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ we can define the classes \mathcal{L}^α and \mathcal{A} . The elements of \mathcal{H}^ω can be distinguished in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ as those, which contain a set in \mathcal{A} . From [14, Prop. 4.6] there is a bijection ϕ of the point set of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ which preserves classes \mathcal{L}^α , \mathcal{A} , and the adjacency \sim^ω w.r.t. the class \mathcal{L}^ω , but it does not preserve the class \mathcal{L}^ω . Consequently, ϕ preserves the adjacency of \mathfrak{A} and therefore it preserves the class of maximal cliques of the adjacency. These cliques are the elements of $\mathcal{H}^\alpha \cup \mathcal{P}^\alpha \cup \mathcal{H}^\omega \cup \mathcal{P}^\omega$. Since \mathcal{A} are preserved and the sort of adjacency is preserved as well, ϕ preserves the classes \mathcal{H}^ω and \mathcal{P}^α . Consequently, $\phi \in \text{Aut}(\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W))$ and thus \mathcal{L}^ω are not definable in $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$.

For the structure $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ the reasoning goes dually. \square

3.2. Specific axioms

The projective Veblen axiom is the fundamental one in projective geometry: a linear space (with lines of size ≥ 3) is a projective space iff it satisfies this axiom. Following this idea we can say that a partial linear space is *projective (in spirit)* iff it satisfies the projective Veblen Condition. Analogously, a linear space with parallelism is an affine space iff it satisfies the affine Veblen Condition (Tamaschke Bedingung) and the Parallelogram Completion Condition. In analogy, we can say that a partial linear space with a partial parallelism is *affine (in spirit)* iff it satisfies these two axioms. The geometry of \mathfrak{A} has a somewhat mixed nature: it has both affine lines and projective lines, so the projective Veblen axiom does not hold in it.

From this point of view \mathfrak{N}^ω , which can be considered as a “projective closure” of \mathfrak{A}^ω , and \mathfrak{N}^α , which can be considered as a “projective closure” of the structure \mathfrak{A}^α , behave differently.

Proposition 3.5. *Each of the four structures \mathfrak{A}^* , \mathfrak{N}^α , \mathfrak{N}^ω , and $\tilde{\mathfrak{A}}$ satisfies the projective Veblen Condition.*

Proof. It is clear that the projective Veblen Condition holds in \mathfrak{A}^* . For the remaining three structures it suffices for an argument to note that the point set \mathcal{F} of each of the structures contains all the points at infinity of \mathfrak{A} , and thus every two coplanar lines intersect each other. \square

Finally, following [18] we have

Fact 3.6. (See [18, Prop. 3.2].) *The structure \mathfrak{A}^τ satisfies the affine Veblen Condition and the Parallelogram Completion Condition.*

Therefore, a spine space determines both structures that are projective and that are affine. The point is, and this will be proved in this paper, that these structures are equivalent to the underlying spine space.

4. Primitive notions

Let us recall the classical result that concerns the geometry of spaces of pencils.

Fact 4.1. (See [5, 7].) *Let $\mathfrak{P} = \mathbf{P}_k(\mathbb{V})$ be not a linear space (that is (iii) in Table 1 holds). The structure \mathfrak{P} is definable in terms of its adjacency. Consequently, the geometry of spaces of pencils can be formulated in the language of binary adjacency of points.*

The above has a well-known algebraic counterpart, commonly referred to as the Chow theorem: the four classes of bijections of $\text{Sub}_k(\mathbb{V})$: preserving the adjacency, preserving pencils, preserving tops, and preserving stars, coincide whenever $2k \neq \dim(\mathbb{V})$ (see [7] for an analytical characterization of such transformations).

Analogous question concerning geometries defined on the point set of a spine space: $\mathbf{A}_{k,m}(\mathbb{V}, W)$ was discussed in [14]. The structure in question is definable in terms of its adjacencies under specific, rather technical assumptions (cf. [14, Prop. 4.5, Lem. 4.8, Cor. 4.11] for more details), which expressed in the language of geometry state that

$$\mathcal{L}^\sigma = \emptyset \quad \text{or} \quad \mathcal{P}^\sigma \neq \emptyset \quad \text{for both } \sigma = \alpha, \omega. \quad (11)$$

So, finally

Fact 4.2. (See [14, Cor. 4.11].) *Under the assumptions (11) the geometry of spine spaces can be developed in the language with the adjacency of points as a single primitive notion.*

In what follows, we shall look for other languages suitable for the geometry of spine spaces.

For any binary relation ρ defined on a set X and $Y \subseteq X$ we put

$$[Y]_\rho = \{x \in X: \forall y \in Y \ x \rho y\}.$$

We write $\{[y_1, \dots, y_n]\}_\rho$ briefly as $[y_1, \dots, y_n]_\rho$. For a relation ρ and any natural n we introduce the relation Δ_n^ρ by the condition

$$\Delta_n^\rho(a_1, \dots, a_n) \Leftrightarrow \neq(a_1, \dots, a_n) \wedge \rho(a_1, \dots, a_n) \wedge \forall b_1, b_2 [b_1, b_2 \rho a_1, \dots, a_n \Rightarrow b_1 \rho b_2]. \quad (12)$$

Definition (12) can be read as follows (assume that ρ is symmetric): $[a_1, \dots, a_n]_\rho$ is a clique w.r.t. ρ . In our investigations on the adjacency of points of a spine space (cf. [14]) the crucial role was played by the relation Δ_3^\sim . Even earlier, as sets $[U_1, U_2, U_3]^\sim$ with $\Delta_3^\sim(U_1, U_2, U_3)$ defined on $\mathfrak{P} = \mathbf{P}_k(\mathbb{V})$ are exactly all the elements of $\mathcal{T} \cup \mathcal{S}$, and the formula $\sim(U_1, U_2, U_3) \wedge \neg \Delta_3^\sim(U_1, U_2, U_3)$ characterizes the collinearity $\mathbf{L}(U_1, U_2, U_3)$ of points of \mathfrak{P} . It is, in fact, a standard way of proving 4.1.

4.1. Line intersection

Let us begin with the relation of adjacency \sim of lines of spine spaces.

Note that in a spine space, not every two crossing lines determine a plane. Let us write

$$\pi(L_1, L_2) \quad \text{iff} \quad L_1 \sim L_2 \text{ and there is a plane, which contains } L_1, L_2,$$

for any two distinct lines L_1, L_2 .

Observation 4.3. The following equivalence holds in \mathfrak{A} .

$$\pi(L_1, L_2) \Leftrightarrow L_1 \sim L_2 \wedge L_1 \neq L_2 \wedge \neg \Delta_2^\sim(L_1, L_2). \quad (13)$$

(In a less elementary way the above property can be expressed as follows: $L_1 \sim L_2$, but $[L_1, L_2]_\sim$ is not a clique.)

Proof. Let U be a common point of L_1, L_2 ; suppose that $\Delta_2^\sim(L_1, L_2)$ does not hold. Let L_3, L_4 be as required in (12). If both L_3, L_4 pass through U , then $L_3 \sim L_4$, and thus either L_1, L_2, L_3 or L_1, L_2, L_4 yield a triangle. Hence L_1, L_2 lie in the plane spanned by this triangle. The converse implication is obvious. \square

On the other hand we see that through every point of \mathfrak{A} there pass at least two noncoplanar lines (cf. (iii) in Table 1). A consequence of that and of 4.3 is

Proposition 4.4. The formula

$$\mathbf{p}(L_1, L_2, L_3) \Leftrightarrow \exists M_1, M_2 [\Delta_2^\sim(M_1, M_2) \wedge M_1, M_2 \sim L_1, L_2, L_3] \quad (14)$$

defines concurrency of lines in terms of their adjacency in the spine space \mathfrak{A} .

Corollary 4.5. The relation of line intersection on \mathcal{L} as well as on $\mathcal{L}^\alpha \cup \mathcal{L}^\omega$ can be used as a primitive notion in the geometry of spine spaces.

Also in structures \mathfrak{A}^α , \mathfrak{A}^ω , and \mathfrak{A}^* through every point there pass at least two noncoplanar lines, and 4.3 is valid in all of them too. Hence we have

Remark 4.6. The statement of 4.4 with \mathfrak{A} replaced by \mathfrak{A}^α , \mathfrak{A}^ω , or \mathfrak{A}^* is true.

4.2. Partial projective closure and its point adjacency

Every line $L = \mathbf{p}(H, B)$ of a space of pencils, provided it is not a linear space, has exactly two extensions to maximal strong subspaces $S(L) = S(H)$ and $T(L) = T(B)$. We write $S_i(L) = S(L) \cap \mathcal{F}_i$ and $T_i(L) = T(L) \cap \mathcal{F}_i$. Recall that the adjacency of points of \mathfrak{N}^σ is

$$\sim = \sim_0^\tau \cup \sim_0^\sigma \cup \sim_1^{-\sigma}$$

where \sim_i^ρ stands for ρ -adjacency determined by lines in \mathcal{L}_i^ρ . Note that \sim_i^τ is defined on $\mathcal{F}_0 \cup \mathcal{F}_1$, while \sim_i^σ is defined on \mathcal{F}_i . We assume in this section that

$$\mathcal{L}_0^\sigma \neq \emptyset \neq \mathcal{L}_1^{-\sigma} \quad \text{and} \quad \mathcal{P}_0^\sigma \neq \emptyset \quad \text{for some } \sigma \in \{\alpha, \omega\}. \quad (15)$$

Let us begin our analysis with

$$\sigma = \omega,$$

so we deal with the three adjacencies \sim_0^τ , \sim_0^ω and \sim_1^α . The following is crucial. Let $L \in \widetilde{\mathcal{L}}^\omega$.

$L \in \mathcal{L}_0^\omega$: $T(L) = T_0(L) \cup T_1(L)$, $T_0(L) \in \mathcal{H}_0^\omega$, $T_1(L) \in \mathcal{P}_1^\alpha$ or – when $\mathcal{P}_1^\alpha = \emptyset$ – $T_1(L) \in \mathcal{L}_1^\alpha$. Further, $S_0(L) \in \mathcal{P}_0^\omega$ and $S_1(L) = \emptyset$. Both $T(L)$ and $S_0(L)$ are \sim -cliques.

$L \in \mathcal{L}_0^\tau$: $T(L)$ is as in the above case. $S(L) = S_0(L) \cup S_1(L)$, $S_0(L) \in \mathcal{H}_0^\alpha$, and $S_1(L) \in \mathcal{P}_1^\omega$, or $S_1(L) \in \mathcal{L}_1^\omega$, or – when $\mathcal{L}_1^\omega = \emptyset$ – $S_1(L)$ is a point and then $S_1(L) \subset L$.

$L \in \mathcal{L}_1^\alpha$: $T(L)$ is as in the first case; if $T_1(L) \in \mathcal{L}_1^\alpha$ then $T_1(L) = L$. $S_0(L) = \emptyset$, and $S_1(L) \in \mathcal{H}_1^\alpha$.

From the above in particular we obtain that for every $U_1, U_2 \in \mathcal{F}$ with $U_1 \neq U_2$ and $U_1 \sim U_2$ we have $\neg \Delta_2^\sim(U_1, U_2)$. Even more can be said, after careful analysis of the above list. For convenience we introduce one more auxiliary relation $\underline{\Delta}^3$:

$$\begin{aligned} \underline{\Delta}(U_1, U_2, U_0) &\Leftrightarrow \Delta_3^\sim(U_1, U_2, U_0) \quad (\text{so } [U_1, U_2, U_0]_\sim \text{ is a } \sim\text{-clique}) \\ &\wedge [[U_1, U_2, U_0]_\sim \setminus \{U_0\}]_\sim \text{ is a } \sim\text{-clique as well.} \end{aligned} \quad (16)$$

³ The definition of $\underline{\Delta}$ can be written in a more elementary way as follows:

$$\underline{\Delta}(U_1, U_2, U_0) \Leftrightarrow \Delta_3^\sim(U_1, U_2, U_0) \wedge \forall U', U'' [\forall U [U \sim U_1, U_2, U_0 \wedge U \neq U_0 \Rightarrow U \sim U', U''] \Rightarrow U' \sim U'']$$

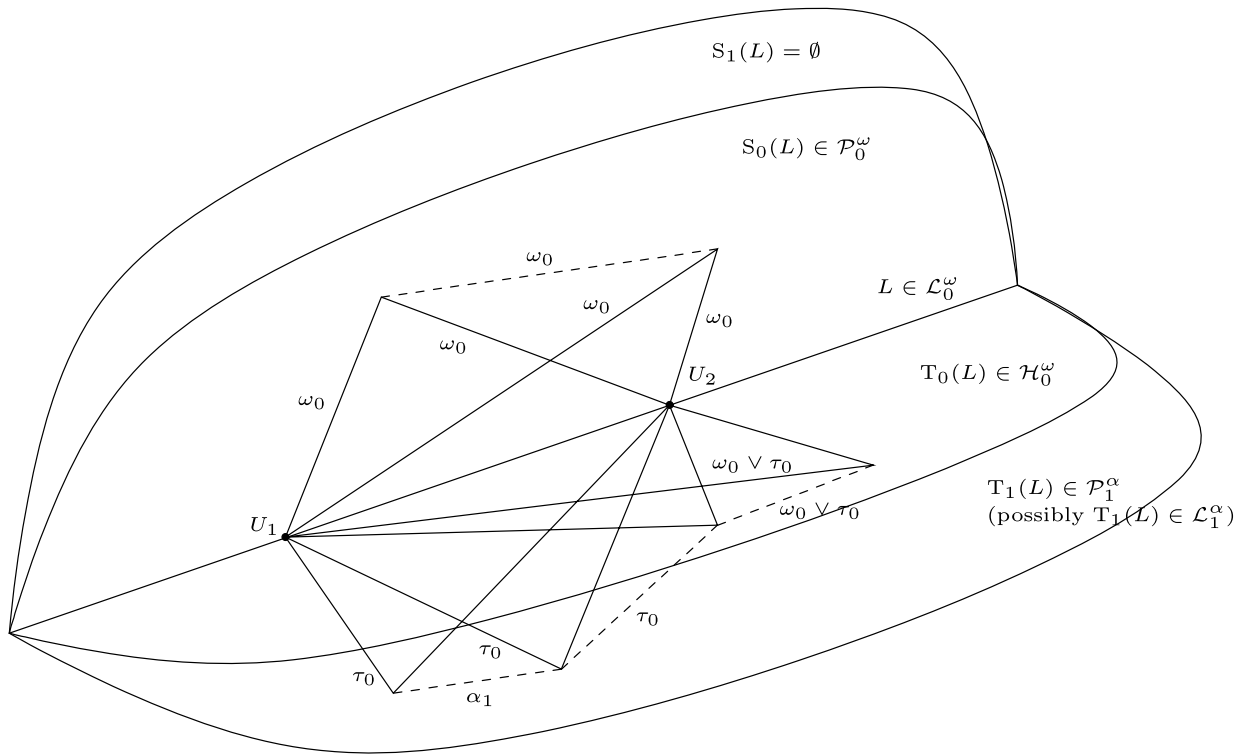


Fig. 1. The structure of $[U_1, U_2]_{\sim}$ for $U_1, U_2 \in \mathcal{F}_0$ on a ω_0 -line L .

Now, let us analyse schemes of possible types of connections between points in $S(L) \cup T(L)$ visualized in Figs. 1–4, and note some technical observations, essential to prove 4.8.

Observation 4.7. Let $U_1, U_2 \in \mathcal{F}$, $U_1 \neq U_2$, and let $U_1, U_2 \in L \in \tilde{\mathcal{L}}^\omega$. Next, let $\mathcal{X} = [U_1, U_2]_{\sim}$ and $U_0 \in \mathcal{F}$ be arbitrary. Clearly, $\mathcal{X} \subseteq T(L) \cup S(L)$.

- (i) If $U_0 \in L$ then $\Delta_3^{\sim}(U_1, U_2, U_0)$ does not hold.
- (ii) Let $L \in \mathcal{L}_0^\omega$. Then $\mathcal{X} = T(L) \cup S_0(L)$ (see Fig. 1).
The relation $\Delta_3^{\sim}(U_1, U_2, U_0)$ holds iff $U_0 \in S_0(L) \setminus L$ or $U_0 \in T(L) \setminus L$ and then $\underline{\Delta}(U_1, U_2, U_0)$ holds as well.
- (iii) Let $L \in \mathcal{L}_0^\omega$ and let $U_1, U_2 \in \mathcal{F}_0$. Let $U_3 \in L \cap \mathcal{F}_1$ and thus $U_3 \in S_1(L)$. Then $T(L) \cup S_1(L) \subset \mathcal{X}$. Let $Y_1, Y_2 \in S_1(L)$; then $Y_1 \neq Y_2$ implies $\neg(Y_1 \sim Y_2)$. In particular, if $Y \in S_1(L) \setminus L$ then $\neg(Y \sim U_3)$ but $Y \sim U$ for every $U \in S_0(L)$. Moreover, $S_0(L) \cap \mathcal{X}$ consists of points of the maximal affine subspace of the slit space $S_0(L)$ containing L (equivalently: containing U_1) and thus $\mathcal{X} \cap S_0(L)$ is a \sim -clique (see Fig. 2).
We have $\Delta_3^{\sim}(U_1, U_2, U_0)$ iff one of the following holds: $U_0 \in T(L) \setminus L$, $U_0 \in S_0(L) \setminus L$ and $S_1(L) \subset L$, or $U_0 \in S_1(L) \setminus L$. In the first and second cases we have $\underline{\Delta}(U_1, U_2, U_0)$ as well, while in the third one $[U_1, U_2, U_0]_{\sim} \setminus \{U_0\} \subset S_0(L)$ and therefore, since $S_1(L)$ is at least a line, $\underline{\Delta}(U_1, U_2, U_0)$ does not hold.
- (iv) Let $L \in \mathcal{L}_0^\tau$, $U_1 \in \mathcal{F}_0$, and $U_2 \in \mathcal{F}_1$. Then $T(L) \subset \mathcal{X}$, and $\mathcal{X} \cap S_1(L) = \{U_2\}$. Finally, $\mathcal{X} \cap S_0(L)$ can be characterized as in (iii). Moreover, \mathcal{X} is the union of two \sim -cliques: $\mathcal{X} \cap S(L)$ and $\mathcal{X} \cap T(L)$ (see Fig. 3).
The relation $\Delta_3^{\sim}(U_1, U_2, U_0)$ holds iff $U_0 \in T(L) \setminus L$ or $U_0 \in S_0(L) \setminus L$. In both cases the relation $\underline{\Delta}(U_1, U_2, U_0)$ holds.
- (v) Let $L \in \mathcal{L}_1^\alpha$. Then $T(L) \subset \mathcal{X}$. The set $\mathcal{X} \cap S_1(L)$ has an irregular structure, as it consists of points of a slit space which form a triangle with U_1, U_2 as one of its edges such that all its sides are projective (see Fig. 4).
The relation $\Delta_3^{\sim}(U_1, U_2, U_0)$ holds iff $U_0 \in T(L) \setminus L$ and then also the relation $\underline{\Delta}(U_1, U_2, U_0)$ holds.

Mutatis mutandis, conditions symmetric to those of 4.7 (with symbols in the following pairs: (α, ω) , (S, T) interchanged) are valid in \mathfrak{N}^α .

We conclude with the following

Proposition 4.8. Let $\sigma \in \{\omega, \alpha\}$. Under the assumptions (15) the collinearity relation \mathbf{L} of \mathfrak{N}^σ can be characterized by the following condition:

$$\mathbf{L}(U_1, U_2, U_3) \Leftrightarrow \sim(U_1, U_2, U_3) \wedge \exists Y_1, Y_2 [U_1, U_2, U_3 \sim Y_1, Y_2 \wedge \neg(Y_1 \sim Y_2) \wedge \underline{\Delta}(U_1, U_2, Y_1)]. \quad (17)$$

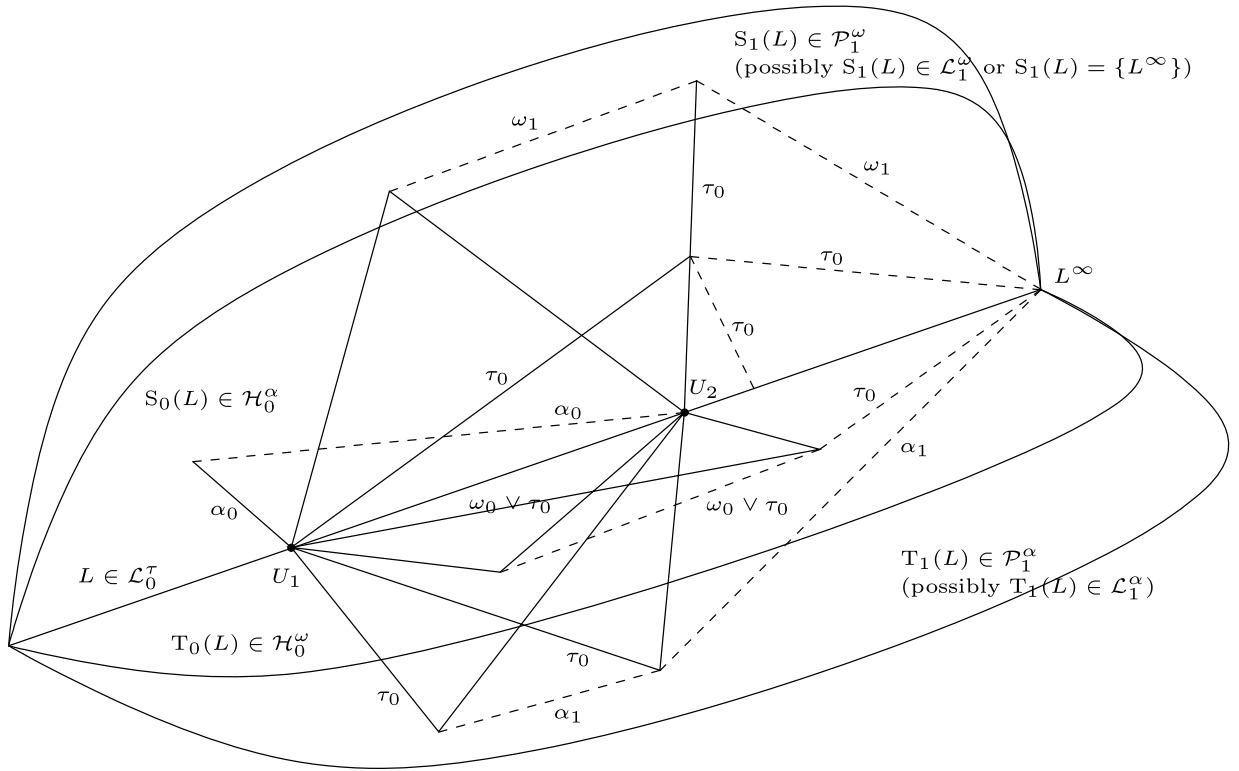


Fig. 2. The structure of $[U_1, U_2]_{\sim}$ for $U_1, U_2 \in \mathcal{F}_0$ on a τ_0 -line L .

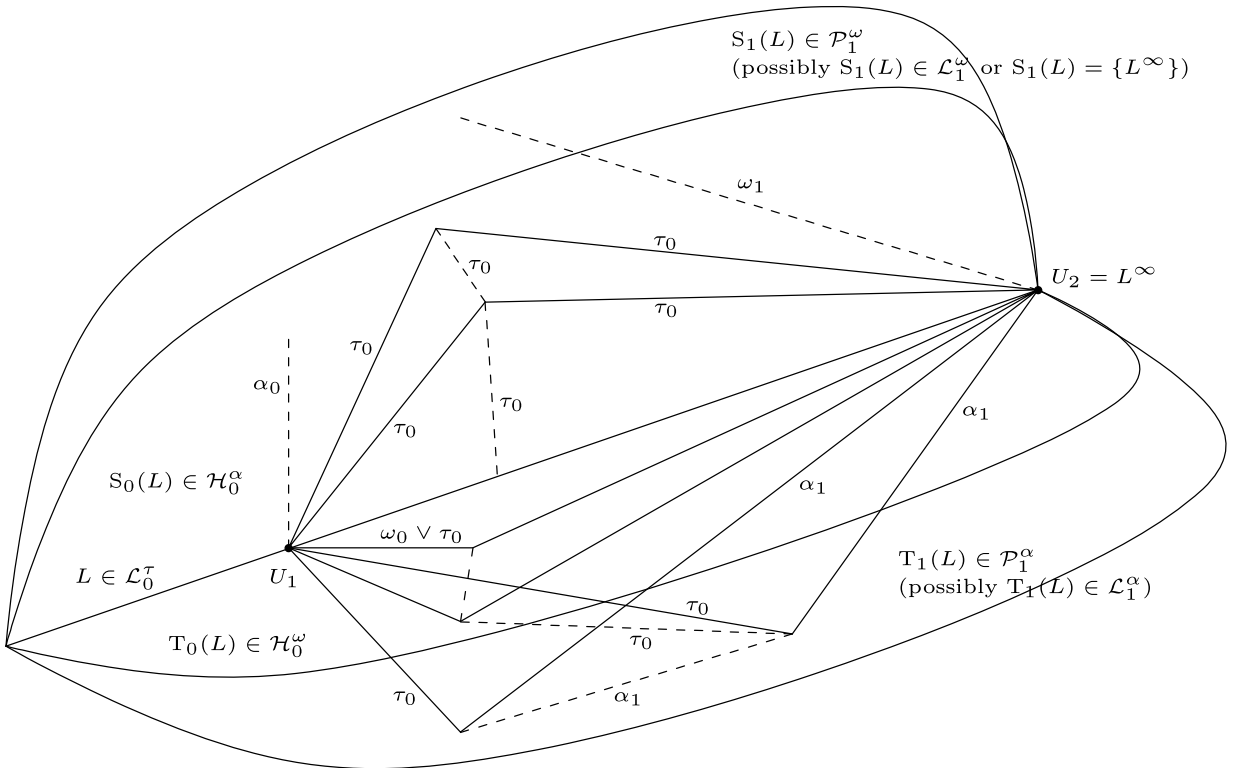


Fig. 3. The structure of $[U_1, U_2]_{\sim}$ for $U_1 \in \mathcal{F}_0$, $U_2 \in \mathcal{F}_1$ on a τ_0 -line L .

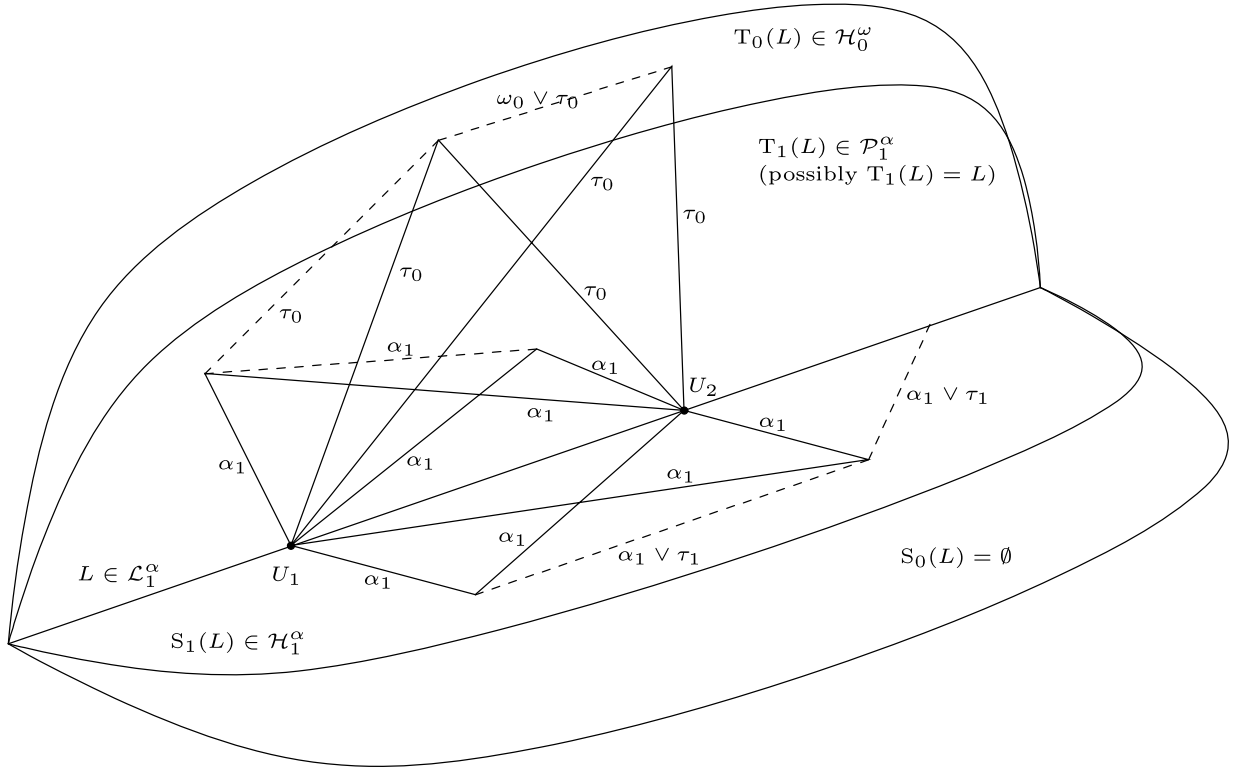


Fig. 4. The structure of $[U_1, U_2]_{\sim}$ for $U_1, U_2 \in \mathcal{F}_1$ on a α_1 -line L .

Proof. Let us take $\sigma = \omega$. The first statement is a nearly immediate consequence of 4.7. To prove \Rightarrow of (17) it suffices to take any $Y_1 \in T_0(L) \setminus L$ and $Y_2 \in S_i(L) \setminus L$ ($i = 0, 1$ respectively). To prove \Leftarrow note first that $\neg \Delta_3^{\sim}(U_1, U_2, U_3)$ and $Y_1 \notin L$ but $Y_1, Y_2 \in [U_1, U_2]_{\sim}$ so, $Y_1 \in T(L) \cup S(L)$. In any case $U_3 \in [U_1, U_2, Y_1]_{\sim} \cap [U_1, U_2, Y_2]_{\sim}$. If $Y_1 \in T(L) \setminus L$ then $Y_2 \in S(L) \setminus L$ and then, obviously, $U_3 \in L$. Let $Y_1 \in S(L) \setminus L$. Let us have a look at the cases of 4.7. In every one of them the condition $\neg(Y_1 \sim Y_2)$ gives $Y_2 \in T(L) \setminus L$ and then, with the standard reasoning we come to $U_3 \in L$.

For $\sigma = \alpha$ the reasoning goes dually. \square

Corollary 4.9. Let $\sigma \in \{\omega, \alpha\}$. Under the assumptions (15) the structure \mathfrak{N}^{σ} is definable in terms of the point adjacency of \mathfrak{N}^{σ} .

Proposition 4.10. Let $\sigma \in \{\omega, \alpha\}$. If $\mathcal{L}_1^{\sigma} \neq \emptyset$, then the classification of lines of \mathfrak{N}^{σ} into three classes \mathcal{L}_0^{σ} , \mathcal{L}_0^{τ} , and $\mathcal{L}_1^{-\sigma}$ can be defined within the incidence structure \mathfrak{N}^{σ} .

Proof. Let L_1, L_2, L_3 be the sides of a triangle in \mathfrak{N}^{σ} and let A be the plane spanned by this triangle. The plane A is in \mathfrak{A} or in \mathfrak{A}^{∞} . Let us introduce some geometrical condition:

(*) on every side L_i of the triangle there is exactly one point U_i of \mathfrak{N}^{σ} such that distinct U_j, U_i cannot be joined in \mathfrak{N}^{σ} .

Considering the sort of L_1, L_2, L_3 , up to an order of indices, we have the following possibilities (partly explained in [18]):

$L_1, L_2, L_3 \in \mathcal{L}_0^{\tau}$: A is either σ -affine and (*) does not hold or A is $(-\sigma)$ -affine and then (*) holds (take the improper point of L_i as the U_i).

$L_1, L_2, L_3 \in \mathcal{L}_0^{\sigma}$: A is either σ -projective or σ -semiaffine and (*) is not valid.

$L_1, L_2 \in \mathcal{L}_0^{\tau}, L_3 \in \mathcal{L}_1^{-\sigma}$: A is σ -affine and (*) does not hold.

$L_1, L_2 \in \mathcal{L}_0^{\sigma}, L_3 \in \mathcal{L}_0^{\tau}$: A is σ -semiaffine and again (*) does not hold.

$L_1, L_2, L_3 \in \mathcal{L}_1^{-\sigma}$: A is either $(-\sigma)$ -projective in \mathfrak{A}^{∞} or $(-\sigma)$ -semiaffine in \mathfrak{A}^{∞} . In both cases (*) does not hold.

If $\mathcal{L}_1^{\sigma} \neq \emptyset$, then for every affine line L_1 there is a $(-\sigma)$ -affine plane A which contains L_1 and therefore L_1 can be completed to a triangle such that (*) holds. Therefore, taking into account what we have found about triangles in \mathfrak{N}^{σ} , this condition distinguishes the class \mathcal{L}_0^{τ} in \mathcal{L}^{σ} in terms of the geometry of \mathfrak{N}^{σ} only. To characterize the lines in $\mathcal{L}_1^{-\sigma}$ observe

that there is only one triangle whose two sides are affine and the remaining one is nonaffine. This nonaffine line is exactly in $\mathcal{L}_1^{-\sigma}$. Finally, the lines that are not in $\mathcal{L}_0^{\tau} \cup \mathcal{L}_1^{-\sigma}$ are the elements of \mathcal{L}_0^{σ} . \square

Corollary 4.11. *Let $\sigma \in \{\omega, \alpha\}$. If $\mathcal{L}_1^{\sigma} \neq \emptyset$, then \mathfrak{A}^{τ} is definable in \mathfrak{N}^{σ} .*

Proof. From 4.10, the families \mathcal{L}_0^{τ} , \mathcal{L}_0^{σ} , and $\mathcal{L}_1^{-\sigma}$ are definable in \mathfrak{N}^{σ} . Recall that $\mathcal{L}_0^{\sigma} \neq \emptyset$ or $\mathcal{L}_1^{-\sigma} \neq \emptyset$. If $\mathcal{L}_0^{\sigma} \neq \emptyset$ we can write: $U \in \mathcal{F}_0$ iff $U \in L$ for some $L \in \mathcal{L}_0^{\sigma}$; if $\mathcal{L}_1^{-\sigma} \neq \emptyset$ we can write: $U \in \mathcal{F}_1$ iff $U \in L$ for some $L \in \mathcal{L}_1^{-\sigma}$. Any way, the point set \mathcal{F}_0 of \mathfrak{A}^{τ} and the set \mathcal{F}_1 are definable in \mathfrak{N}^{σ} . We can identify the elements of \mathcal{A} with the elements of \mathcal{L}_0^{τ} so, the line set of \mathfrak{A}^{τ} is definable in \mathfrak{N}^{σ} . Finally, for $L_1, L_2 \in \mathcal{L}_0^{\tau}$ we have $L_1 \parallel L_2$ iff there is $U \in \mathcal{F}_1$ such that $U \in L_1, L_2$. \square

4.3. The role of parallelism – the structure \mathfrak{A}^{τ}

To close this part we need to characterize the geometry of \mathfrak{A} in terms of the geometry of \mathfrak{A}^{τ} , using some techniques related to \mathfrak{N}^{σ} .

Trivially, \mathcal{F}_1 can be interpreted in terms of \mathfrak{A}^{τ} , as a point $Y \in \mathcal{F}_1$ uniquely corresponds to the equivalence class $[L]_{\parallel} \in \mathcal{A}/\parallel$, where $L \in \mathcal{A}$ and $L \cup \{Y\} \in \mathcal{L}_0^{\tau}$. Consequently, we can define in \mathfrak{A}^{τ} the elements of \mathcal{L}_0^{τ} .

For brevity we write $U_1 \sim^{\alpha \vee \sigma_2} U_2$ which means that $U_1 \sim^{\sigma_1}$ or $U_1 \sim^{\sigma_2} U_2$, where $\sigma_1, \sigma_2 \in \{\alpha, \omega, \tau\}$.

Let us note that if $\mathcal{L}_1^{\alpha} \cup \mathcal{L}_1^{\omega} \neq \emptyset$, then the following formula expressible in terms of \mathfrak{A}^{τ} defines the projective adjacency of \mathcal{F}_1 : for distinct $U_1, U_2 \in \mathcal{F}_1$ we have

$$U_1 \sim_1^{\alpha \vee \omega} U_2 \Leftrightarrow \exists L_1, L_2, M_1, M_2 \in \mathcal{L}_0^{\tau} [L_1 \neq L_2 \wedge M_1 \neq M_2 \\ \wedge U_1 \in L_1, L_2 \wedge U_2 \in M_1, M_2 \wedge (L_i \text{ crosses } M_j \text{ for } i, j = 1, 2)]. \quad (18)$$

Then,⁴ the adjacency relation $\sim_0^{\alpha \vee \omega}$ of \mathfrak{A} can be characterized as follows:

$$U_1 \sim_0^{\alpha \vee \omega} U_2 \Leftrightarrow U_1, U_2 \in \mathcal{F}_0 \wedge \neg (U_1 \sim_0^{\tau} U_2) \\ \wedge \exists Y_1, Y_2 \in \mathcal{F}_1 [Y_1 \neq Y_2 \wedge Y_1 \sim_1^{\omega \vee \alpha} Y_2 \wedge Y_1, Y_2 \sim_0^{\tau} U_1, U_2]. \quad (19)$$

To justify the correctness of the above definition it suffices to analyze the extensions of a projective line $L = \overline{Y_1, Y_2} \in \mathcal{L}_1^{\sigma}$ and make use of the conditions of 4.7 and their dual. For $\sigma = \omega$ we see that $S_0(L) \in \mathcal{H}_0^{\alpha}$, and $T_0(L) = \emptyset$; consequently, if $U_1, U_2 \in [Y_1, Y_2]_{\sim \tau}$, then $U_1 \sim^{\alpha \vee \tau} U_2$. Conversely, if $M = \overline{U_1, U_2} \in \mathcal{L}_0^{\alpha}$, then $S_1(M) \in \mathcal{P}_1^{\omega}$ contains ω -lines. If $\sigma = \alpha$, the reasoning is similar. Thus we proved

Proposition 4.12. *If $\mathcal{L}_1^{\alpha} \cup \mathcal{L}_1^{\omega} \neq \emptyset$, then the adjacency of \mathfrak{A} is definable in terms of the geometry of \mathfrak{A}^{τ} .*

Clearly, if $\mathcal{L}_0^{\alpha} \cup \mathcal{L}_0^{\omega} = \emptyset$, then $\mathfrak{A}^{\tau} = \mathfrak{A}$. Again by Table 1 if $\mathcal{P}^{\sigma} \neq \emptyset$, then $\mathcal{L}_1^{\sigma} \neq \emptyset$. Thus as a consequence of 4.12 and 4.2 we obtain

Corollary 4.13. *Under the assumptions (11) \mathfrak{A} is (re)definable in \mathfrak{A}^{τ} and therefore the structures \mathfrak{A} and \mathfrak{A}^{τ} are definitionally equivalent.*

The result seems interesting for its own sake as well. It is known (see 3.1) that the geometry of a spine space \mathfrak{A} can be considered as a pure incidence theory of its projective lines only (and thus the geometry of \mathfrak{A} can be seen as a kind of “projective” geometry). It is also known that the geometry of the structure $\langle \mathcal{F}_0, \mathcal{A} \rangle$ is essentially weaker than the geometry of \mathfrak{A} . But the theory of parallelism of \mathfrak{A} turns out to be equivalent to the theory of \mathfrak{A} , which makes the geometry of \mathfrak{A} affine in spirit.

Remark 4.14. One can also consider a (possibly slightly artificial) incidence structure $\mathfrak{N}^{\tau} := \langle \mathcal{F}_0 \cup \mathcal{F}_1, \mathcal{L}_0^{\tau} \rangle$ whose lines are (practically) affine lines of \mathfrak{A} . But now, \mathfrak{N}^{τ} is projective in spirit. What is more important, if $\mathcal{L}_1^{\alpha} \cup \mathcal{L}_1^{\omega} \neq \emptyset$, then the structure \mathfrak{A}^{τ} is definable in terms of \mathfrak{N}^{τ} .

Consequently, if $\mathcal{L}_1^{\sigma} \neq \emptyset$ and (11) holds for $-\sigma$ with some $\sigma \in \{\alpha, \omega\}$, then \mathfrak{A} is definable in \mathfrak{N}^{τ} (comp. 3.1(iii)).

Proof. It suffices to note that \mathfrak{N}^{τ} is not a Γ -space (cf. [6]); more specifically, a vertex U_1 of a triangle U_1, U_2, U_3 in \mathfrak{N}^{τ} cannot be joined with a point on $\overline{U_2, U_3}$ iff $U_1 \in \mathcal{F}_1$ (the plane A spanned by U_1, U_2, U_3 is affine in \mathfrak{A} , but its improper points in \mathfrak{N}^{τ} yield a projective line that crosses every other line of A). If \mathfrak{A} contains an affine plane (of any sort), then the above property defines \mathcal{F}_1 in terms of \mathfrak{N}^{τ} and thus \mathfrak{A}^{τ} is definable in \mathfrak{N}^{τ} . The second claim follows by 4.13. \square

⁴ On a side note, to define projective lines on \mathfrak{A}^{∞} consider a parallelogram L_1, L_2, M_1, M_2 in \mathfrak{A} like in (18). A point $U_3 \in \mathcal{F}_1$ is on $\overline{U_1, U_2}$ iff it is on a line $L_3 \in \mathcal{L}_0^{\tau}$, which crosses the lines L_i, M_j required in (18).

In the classical case of affine and projective geometry it is impossible to distinguish completed affine lines from projective lines on the horizon. The case of partial projective closure \mathfrak{N}^σ turns out to be different in that \mathfrak{A} can be recovered from it.

Corollary 4.15. *If $\mathcal{L}_1^\alpha \neq \emptyset \neq \mathcal{L}_1^\omega$ then the spine space \mathfrak{A} can be defined in \mathfrak{N}^σ and the structures \mathfrak{A} and \mathfrak{N}^σ are definitionally equivalent.*

Proof. Since $\mathcal{L}_1^\sigma \neq \emptyset$, we get from 4.11 that the point set \mathcal{F}_0 of \mathfrak{A} is definable from \mathfrak{N}^σ and from 4.10 that the families \mathcal{A} and \mathcal{L}_0^σ are definable as well; consequently, the relations \sim_0^τ and \sim_0^σ are definable. From 4.12, the point adjacency \sim of \mathfrak{A} is definable from \mathfrak{N}^σ and thus the relation $\sim_0^{-\sigma}$ is also definable in \mathfrak{N}^σ . Taking into account the assumption $\mathcal{L}_1^{-\sigma} \neq \emptyset$ and (xii), (xiii) in Table 1 from [14, Prop. 4.5] we have the class $\mathcal{L}_0^{-\sigma}$ definable in \mathfrak{N}^σ . Finally, \mathfrak{A} is definable in \mathfrak{N}^σ . The second claim follows now from 3.2. \square

Corollary 4.16. *Under assumptions of (15) the structure \mathfrak{A} can be defined in terms of point adjacency of \mathfrak{N}^σ .*

Proof. First, from 4.9 we remind that the structure \mathfrak{N}^σ is definable in terms of point adjacency \sim of \mathfrak{N}^σ . Next, we observe that $\mathcal{L}_1^\alpha \neq \emptyset \neq \mathcal{L}_1^\omega$ is a consequence of (15) and thus our claim follows directly from 4.15. \square

Remark 4.17. With methods used in proofs of 4.14, 4.10, and 4.11 we get that if $\mathcal{L}_1^\alpha, \mathcal{L}_1^\omega \neq \emptyset$ then \mathfrak{A} is definable in $\tilde{\mathfrak{A}}$ and, consequently, \mathfrak{A} and $\tilde{\mathfrak{A}}$ are definitionally equivalent.

Proof. Note that \mathfrak{A}^∞ contains nonaffine semiaffine planes exactly when it contains both affine lines and projective lines of some sort σ . On the other hand \mathfrak{A}^∞ does not contain an affine line only in the case when it is a space of pencils, which is equivalent to the disjunction $\mathcal{L}_1^\alpha = \emptyset$ or $\mathcal{L}_1^\omega = \emptyset$ (comp. (iii), (xii), (xiii) in Table 1).

The structure $\tilde{\mathfrak{A}}$ is not a Γ -space. More precisely, continuing analysis of 4.10 of possible triangles, now considered in the whole structure $\tilde{\mathfrak{A}}$, we see that a vertex U_1 of a triangle U_1, U_2, U_3 of $\tilde{\mathfrak{A}}$ cannot be joined with a point on $\overline{U_2, U_3}$ iff the plane A spanned by U_1, U_2, U_3 is semiaffine in \mathfrak{A}^∞ (note: for $U_1, U_2, U_3 \in \mathcal{F}_1$ the sides of the triangle must be projective and thus A cannot be affine!). From the assumption every point in \mathcal{F}_1 can be completed to such a triangle and thus the set \mathcal{F}_1 is definable in $\tilde{\mathfrak{A}}$. Now it is clear that also the set \mathcal{F}_0 and the families \mathcal{L}_0^τ and \mathcal{L}_0^ω are definable in $\tilde{\mathfrak{A}}$, which closes the proof. \square

4.4. Star and top intersection

It is immediate from 2.1 that the incidence structures of \mathfrak{A} and $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ and $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ are mutually definable if the point adjacency of \mathfrak{A} suffices to recover its lines. When we deal with the adjacency of lines of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ (of $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$) we must exclude some degenerate cases. Namely, for $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ we must assume that $k > 2$, since the structure $\mathbf{A}_{2,m}^{\text{star}}(\mathbb{V}, W)$ is a dual linear space and thus its line adjacency is total. Analogously, dealing with the line adjacency of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ one should assume that $k < \dim(\mathbb{V}) - 2$.

The map $S: H \mapsto S(H)$ establishes a 1–1 correspondence between the points of \mathfrak{N}^ω and the lines of $\mathbf{A}_{k+1,m+1}^{\text{star}}(\mathbb{V}, W)$, and the map $T: B \mapsto T(B)$ establishes a 1–1 correspondence between the points of \mathfrak{N}^α and the lines of $\mathbf{A}_{k-1,m}^{\text{top}}(\mathbb{V}, W)$. These identifications can be used to prove

Proposition 4.18. *If $\mathcal{L}^\alpha, \mathcal{P}_1^\omega$, and \mathcal{P}_2^α ($\mathcal{L}^\omega, \mathcal{P}_1^\alpha$, and \mathcal{P}_2^ω) are nonvoid, then the structure $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ ($\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ resp.) can be defined in terms of the adjacency of its lines.*

Proof. Set $\mathfrak{A}' := \mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$. From 2.2, the structure of adjacency of lines in $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$ corresponds under the map S^{-1} to the structure of adjacency of points of \mathfrak{N}^ω defined over the spine space $\mathfrak{A} = \mathbf{A}_{k,m}(\mathbb{V}, W)$ with $k = k' - 1$ and $m = m' - 1$. Under suitable dimension assumptions, in terms of this point adjacency one can define (cf. 4.8) the lines of \mathfrak{N}^ω and, after that, one can define the structure \mathfrak{A} within \mathfrak{N}^ω .

To go further we must analyse deeper the connections between $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$ and \mathfrak{N}^ω . We intend to (re)define the notion of a point of $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$ in terms of suitable line adjacency; this means we are going to define families of the form $T^*(U) := \{S(H) : U \in S(H)\}$ with $U \in \mathcal{F}_{k',m'}(W) = \mathcal{F}_{k+1,m+1}(W)$, which are images under S of the sets of the form $\{H \in \mathcal{F}_{k,m} \cup \mathcal{F}_{k,m+1} : H \subset U\}$. Observe that the set $T_0(U) := \{H \in \mathcal{F}_{k,m} : H \subset U\}$ is an ω -top in \mathfrak{A} , i.e. an element of \mathcal{H}_0^ω . The set $T_1(U) := \{H \in \mathcal{F}_{k,m+1} : H \subset U\}$ is an α -top in \mathfrak{A}^∞ , i.e. an element of \mathcal{P}_1^ω ; moreover, $T_1(U)$ is the set of improper points (the horizon) of $T_0(U)$. Once we can recover \mathfrak{A} in terms of the point adjacency of \mathfrak{N}^ω , all these families can be defined in terms of this adjacency, as they are strong subspaces of suitable types. Expressing the above in terms of the underlying structure $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$ we obtain that the concurrency of stars and after that the notion of a point of \mathfrak{A}' (that is the notion of a point of $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$) and point-line incidence of $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$ are definable in terms of line adjacency of $\mathbf{A}_{k',m'}^{\text{star}}(\mathbb{V}, W)$.

Analogous reasoning can be applied to $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$. Gathering together respective dimensional assumptions necessary to provide the above reasoning, with the help of Table 1 we obtain our claim. \square

Note that to use the above representation of the lines of $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ via the map S we should assume that $m \geq 1$ (i.e. $\mathcal{L}^\omega \neq \emptyset$), because otherwise the structure of the form \mathfrak{A}^ω representing the lines of $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$ is not well defined (cf. (i) in Table 1). Analogously, to represent the lines of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ via the map T we should assume that $m \geq k - \text{codim}(W) + 1$ (i.e. $\mathcal{L}^\alpha \neq \emptyset$). However, we have:

Proposition 4.19. *Both of the structures $\mathbf{A}_{k,0}^{\text{star}}(\mathbb{V}, W)$ and $\mathbf{A}_{k,k-\text{codim}(W)}^{\text{top}}(\mathbb{V}, W)$ can be defined in terms of their line adjacency.*

Proof. Take $m = 0$; then $\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W) = \langle \mathcal{F}_{k,0}(W), \mathcal{H}^\alpha \rangle$. Via the map S we can identify the lines of $\mathbf{A}_{k,0}^{\text{star}}(\mathbb{V}, W)$ with the points of $\mathfrak{A}' := \mathbf{A}_{k-1,0}(\mathbb{V}, W)$. The line adjacency in $\mathbf{A}_{k,0}^{\text{star}}(\mathbb{V}, W)$ corresponds under S to α -adjacency of points of \mathfrak{A}' . Moreover, \mathfrak{A}' does not contain an ω -projective line and therefore one can define \mathfrak{A}' in terms of its α -adjacency of points. As in the proof of 4.18 we conclude that $\mathbf{A}_{k,0}^{\text{star}}(\mathbb{V}, W)$ can be defined in terms of its line adjacency. Now take $m = k - \text{codim}(W)$; then $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W) = \langle \mathcal{F}_{k,k-\text{codim}(W)}(W), \mathcal{H}^\omega \rangle$ and the reasoning runs dually. \square

As an immediate consequence of 4.18 and 2.1 we obtain

Corollary 4.20. *If \mathcal{L}^α , \mathcal{P}_1^ω , and \mathcal{P}_2^α (\mathcal{L}^ω , \mathcal{P}_1^α , and \mathcal{P}_2^ω) are nonvoid, then the structure \mathfrak{A} can be defined in terms of the adjacency of its tops (stars), i.e. in terms of line adjacency of $\mathbf{A}_{k,m}^{\text{top}}(\mathbb{V}, W)$ ($\mathbf{A}_{k,m}^{\text{star}}(\mathbb{V}, W)$).*

Appendix A

A.1. Classical geometries

A.1.1. Excluded geometries

The class of spine spaces contains also some “classical” geometries which were excluded due to the assumptions (i)–(iv) of Table 1. In the excluded geometries some of the above systems of notions degenerate or loose their sense (e.g. if \mathfrak{A} is a linear space then the point adjacency is total and therefore useless). Let us make several brief comments on the excluded cases.

If \mathfrak{A} is not a linear space but it is a space of pencils (that is if \mathfrak{A} is the Grassmannian of proper $[k-1]$ -subspaces of a projective space, cf. [3]) then point adjacency suffices (cf. 4.1) to express the geometry of \mathfrak{A} ; the line intersection suffices (repeat the reasoning of Section 4.1); incidence structure with stars (with tops) as lines also suffices (the point is to define suitable adjacency and use 4.1). With the Plücker embedding of a space of pencils (cf. [10]) the result concerning line adjacency (adjacency on stars or adjacency on tops) can be viewed as a generalization of known results concerning adjacencies in polar spaces (cf. [6,15]).

If \mathfrak{A} is at least 3-dimensional linear space and it is not a projective space (that is \mathfrak{A} is a slit space), then line intersection suffices to express the geometry of \mathfrak{A} .

Proof. Let \mathfrak{A} be obtained by removing from a projective space \mathfrak{P} its proper subspace \mathcal{W} and let \sim stand for the adjacency of lines of \mathfrak{A} . Consider the set $X = [L_1, L_2, L_3] \sim$ with pairwise adjacent lines L_1, L_2, L_3 . Three possibilities arise:

- L_1, L_2, L_3 have a common point a but do not lie on a plane; then $X = S(a)$ consists of all the lines through a so, $\Delta_3^\sim(L_1, L_2, L_3)$ holds.
- L_1, L_2, L_3 lie on a plane A , but do not have a common point; If A misses \mathcal{W} then $X = T(A)$ consists of all the lines on A and $\Delta_3^\sim(L_1, L_2, L_3)$ holds. If A and \mathcal{W} have common point then $\Delta_3^\sim(L_1, L_2, L_3)$ fails.
- L_1, L_2, L_3 are on a plane A and pass through a point a . There is a line $K_1 \in X$ on A which misses a and a line K_2 through a not on A ; clearly $K_1 \approx K_2$ and thus $\Delta_3^\sim(L_1, L_2, L_3)$ fails.

Write $\mathcal{C} := \{[L_1, L_2, L_3] \sim : \Delta_3^\sim(L_1, L_2, L_3)\}$. To complete the proof it suffices to characterize the sets $S(a)$ in terms of the adjacency of lines. If every plane of \mathfrak{P} touches \mathcal{W} we are done: $\{S(a) : a \text{ a point}\} = \mathcal{C}$. If not, we proceed as follows. Take two distinct $M_1, M_2 \in X \in \mathcal{C}$ and a line $M_3 \notin X$ such that M_1, M_2, M_3 are pairwise adjacent. If $X = S(a)$ for some point a then M_1, M_2, M_3 span a plane A ; one can find M_1, M_2, M_3 such that A crosses \mathcal{W} and thus there are M_1, M_2, M_3 for which $\Delta_3^\sim(M_1, M_2, M_3)$ fails. If $X = T(A)$ for some plane A then M_3 is a line through the common point of M_1, M_2 not on A and thus $\Delta_3^\sim(M_1, M_2, M_3)$ holds. Thus the sets of the form $S(a)$ are distinguished within \mathcal{C} . \square

For at least 3-dimensional affine spaces and for at least 4-dimensional projective spaces the result is well known as it is a result of a search for an adequate system of primitive notions for line geometry (cf. [8,13,12,9,22]).

Our results can be also easily applied to geometries which were our first models of spine geometry.

A.1.2. Affine Grassmannians

If \mathfrak{A}_0 is an affine space defined over a vector space W with the coordinate field \mathfrak{F} then with $\mathbb{V} = W \oplus \mathfrak{F}$ we can represent the affine Grassmannian of $(k-1)$ -subspaces of \mathfrak{A}_0 as the spine space $\mathfrak{A} = \mathbf{A}_{k,k-1}(\mathbb{V}, W)$. To avoid trivial cases we assume that the points of this affine Grassmannian are neither points of \mathfrak{A}_0 ($k-1 \neq 0$) nor hyperplanes in \mathfrak{A}_0 ($k-1 \neq \dim(W)-1$). Trivially, from 4.5

line intersection in the affine Grassmannian (primarily defined as a partial linear space) suffices to characterize the geometry of this Grassmannian.

Observing respective conditions in Table 1 we see that $\mathcal{L}^\alpha = \emptyset \neq \mathcal{L}^\omega$, \mathcal{P}^ω and $\mathcal{L}_1^\alpha = \emptyset \neq \mathcal{L}_1^\omega$. This gives that (11) holds and thus from 4.2 we get that

the point adjacency of \mathfrak{A} suffices to characterize the geometry of \mathfrak{A} , that is the geometry of $(k-1)$ -th affine Grassmannian over \mathfrak{A}_0 can be characterized in terms of adjacency of $(k-1)$ -subspaces of \mathfrak{A}_0 ;

after that with standard methods we get that the geometry of \mathfrak{A}_0 can be characterized in terms of adjacency of its $(k-1)$ -subspaces.

The lines of \mathfrak{A} are affine pencils of subspaces of \mathfrak{A}_0 ; these are either proper pencils or so called parallel pencils, and the latter are exactly those which are affine lines of \mathfrak{A} in the sense adopted in the paper. Parallel pencils have form $\mathbf{p}^*(U_0, B) = \{U : U \text{ is a } (k-1)\text{-subspace of } \mathfrak{A}_0, U_0 \parallel U \subset B\}$, where B is a k -subspace of \mathfrak{A}_0 and $U_0 \parallel B$ is a $(k-1)$ -subspace. One can see that $\mathbf{p}^*(U'_0, B') \parallel \mathbf{p}^*(U''_0, B'')$ holds in \mathfrak{A} iff B', B'' are parallel in \mathfrak{A}_0 . Thus from 4.13 we have that

the geometry of the $(k-1)$ -th affine Grassmannian \mathfrak{A} can be characterized in the language of $(k-1)$ -subspaces, parallel pencils and the parallelism of such pencils.

Note that the condition (15) is false for both $\sigma = \alpha$ and $\sigma = \omega$ and thus the results concerning structures like \mathfrak{N} cannot be applied.

Finally, since $k - \text{codim}(W) = m$ in the case analysed now, from 3.4 and 4.19 we obtain that

the geometry of the $(k-1)$ -th affine Grassmannian can be characterized as an incidence structure with tops only (with stars only),⁵ and it can be also characterized in terms of adjacency of its tops

A.1.3. Structure of linear complements

A structure of linear complements of W is a spine space of the form $\mathfrak{A} = \mathbf{A}_{k,0}(\mathbb{V}, W)$, where W is a subspace of \mathbb{V} with $\text{codim}(W) = k$ (cf. [19,4]; comp. also [1,22]). Assume that $1 \neq k \neq \dim(\mathbb{V}) - 1$, since otherwise \mathfrak{A} is simply an affine space. From respective conditions in Table 1 we get that $\mathcal{L}^\sigma = \emptyset$ and $\mathcal{L}_1^\sigma \neq \emptyset$ for both $\sigma = \alpha, \omega$ and thus (11) holds, while (15) fails. In view of 4.2 and 4.5,

the geometry of the structure of linear complements \mathfrak{A} can be characterized in terms of binary collinearity of points as well as in terms of line intersection.

There is no need to bother about the role of the parallelism in \mathfrak{A} , because in this case it is definable in terms of the lines of \mathfrak{A} (cf. 3.1(v)). Stars and tops of \mathfrak{A} are affine spaces (suitably covering the point set of \mathfrak{A}). From 3.4 and 4.19 we get that

the geometry of the structure of linear complements of W can be characterized as an incidence structure with tops only (with stars only) and it can be also characterized in terms of adjacency of its tops and in terms of adjacency of its stars.

A.2. Chow's view

Instead of saying that: *the structure of \mathfrak{A} can be defined in terms of some relation δ on subspaces of \mathfrak{A}* , or that: *the relation δ is a sole primitive notion for \mathfrak{A}* , we can reformulate most, if not all, of our results in this paper into statements in the vein of the famous theorem of Chow (cf. [5,7]) that would say that: *an automorphism of the relation δ on subspaces of \mathfrak{A} is induced by an automorphism of \mathfrak{A}* . For example, in view of our Main Theorem, under suitable assumptions, bijective transformations that preserve any of the following relations:

- line adjacency of \mathfrak{A} (cf. (ii)),
- adjacency of projective lines of \mathfrak{A} (cf. (iii)),

⁵ Note that these are classes of strong subspaces that were used by Tallini (cf. [21], see also [2]) to axiomatically characterize affine Grassmannians.

- adjacency of stars of \mathfrak{A} (cf. (ix)),
- adjacency of tops of \mathfrak{A} (cf. (x)),

are induced by automorphisms of \mathfrak{A} .

A.3. Conditions and related parameters

Table 1

Conditions and related parameter sets imposed on a spine space $\mathbf{A}_{k,m}(\mathbb{V}, W)$.

	Condition	Parameter set
	Global assumptions:	
(i)	$\mathbf{A}_{k,m}(\mathbb{V}, W)$ is well defined	$0, k - \text{codim}(W) \leq m$
(ii)	$\mathbf{A}_{k,m}(\mathbb{V}, W)$ is not a space of pencils	$m < k, \dim(W)$
(iii)	$\mathbf{A}_{k,m}(\mathbb{V}, W)$ is not a linear space	$1 < k < \dim(\mathbb{V}) - 1$
(iv)	$\mathbf{A}_{k,m}(\mathbb{V}, W)$ is not a line	$3 \leq \dim(\mathbb{V})$
	Specific conditions:	
(v)	$\mathcal{L}^\alpha \neq \emptyset$	$k - \text{codim}(W) < m$
(vi)	$\mathcal{L}^\omega \neq \emptyset$	$0 < m$
(vii)	$\mathcal{A} \neq \emptyset$ ($\mathcal{L}_0^\tau \neq \emptyset$)	always
(viii)	$\mathcal{P}^\alpha \neq \emptyset$	$k - \text{codim}(W) < m < k - 1$
(ix)	$\mathcal{P}^\omega \neq \emptyset$	$0 < m < \dim(W) - 1$
(x)	$\mathcal{H}^\alpha \neq \emptyset$	always
(xi)	$\mathcal{H}^\omega \neq \emptyset$	always
(xii)	$\mathcal{L}_1^\alpha \neq \emptyset$	$m < k - 1$
(xiii)	$\mathcal{L}_1^\omega \neq \emptyset$	$m < \dim(W) - 1$
(xiv)	$\mathcal{P}_1^\alpha \neq \emptyset$	$m < k - 2$
(xv)	$\mathcal{P}_1^\omega \neq \emptyset$	$m < \dim(W) - 2$
(xvi)	$\mathcal{P}_2^\alpha \neq \emptyset$	$m < k - 3$
(xvii)	$\mathcal{P}_2^\omega \neq \emptyset$	$m < \dim(W) - 3$

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